Notes on Baker-Campbell-Hausdorff (BCH) Formulae

The BCH formula for the product of the exponentials of two operators A and B is

$$e^A e^B = e^{A+B+[A,B]/2+...}\,, \tag{1}$$

where the ... represents terms that are at least cubic in A and B and involve nested commutators of A and B. The formula is important in the theory of Lie Groups as well as quantum mechanics. An important special case where an exact formula exists is

$$e^A e^B \;=\; e^{A+B+[A,B]/2} \qquad [A,B]=c \tag{2}$$

where c is a c-number (or $[c,A]=[c,B]=0$). An important physics application of this form of the BCH formulae is in the theory of coherent states. In general, there is no closed form expression for the exponent on the r.h.s. of Eq. (1).

Here we will develop a systematic method for computing the higher order terms in the exponent of Eq. (1). This is interesting in its own right and additional useful formulae will be derived in the process of obtaining this result. One of these equations is

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + ...\frac{1}{n!}[\overbrace{A,[A,...[A}^{n\,A's},B]..] + ...\,. \tag{3}$$

To prove this, define an operator-valued function of the c-number x:

$$\begin{aligned} F(x) &= e^{xA} B e^{-xA} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} F_n x^n\,. \end{aligned} \tag{4}$$

The formulae in Eq. (3) corresponds to $F(1)$ and in the second line $F(x)$ is written as contains a Taylor series in x. The reason for this is that the derivative of $F(x)$ can be calculated to obtain a recursion relation for the coefficients F_n. Once this recursion relation is solved, $F(x)$ is known and then setting $x=1$ leads to Eq. (3). It should be clear from the first line of Eq. (4) that

$$\frac{d}{dx}F(x) = [A, F(x)]\,. \tag{5}$$

Using the series form for $F(x)$ in this expression we find

$$\sum_{n=1}^{\infty} F_n \frac{1}{(n-1)!} x^{n-1} = \sum_{n=0} \frac{1}{n!}[A,F_n] x^n\,. \tag{6}$$

Shifting $n \to n+1$ on the l.h.s. of this equation and equating terms with the same power of x we find $F_{n+1} = [A, F_n]$. It is clear that $F_0 = B$ and the remaining F_n follow from this recursion relation. The result in Eq. (3) follows upon setting $x=1$.

A second useful formulae is

$$\frac{d}{dx} e^{A(x)} = \int_0^1 dy\, e^{(1-y)A} \frac{dA}{dx} e^{yA}\,. \tag{7}$$

Here $A(x)$ is an arbitrary function of x. To see why Eq. (7) is true both sides must be expanded in a power series in A and $A' = dA/dx$. On the l.h.s. we find

$$\begin{aligned} \frac{d}{dx}e^{A(x)} &= \frac{d}{dx}\left(1 + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + ...\right) \\ &= A' + \frac{A'A + AA'}{2} + \frac{A'A^2 + AA'A + A^2A'}{3!} + \dots . \end{aligned} \tag{8}$$

The key point is that when we differentiate A^n we cannot assume that A and A' commute so we get n terms with $n-1$ A's and one A' with all possible orderings of these n factors. This series can written as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+m+1)!}A^n A' A^m \tag{9}$$

However this is the series generated by the integral in Eq. (7). Expanding both exponentials one finds

$$\begin{aligned} \int_0^1 dy\, e^{(1-y)A}\frac{dA}{dx}e^{yA} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{A^n A' A^m}{n!m!}\int_0^1 dy(1-y)^n y^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(n+m+1)!}A^n A' A^m \,, \end{aligned} \tag{10}$$

where the last line follows from the integral

$$\int_0^1 dy(1-y)^n y^m = \frac{n!m!}{(n+m+1)!} \,. \tag{11}$$

Finally multiply Eq. (7) by e^{-A} to obtain

$$\begin{aligned} e^{-A}\frac{d}{dx}e^{A} &= \int_0^1 dy\, e^{-yA}\frac{dA}{dx}e^{yA} \,. \\ &= A' + \frac{1}{2}[A', A] + \frac{1}{3!}[[A', A], A] + ... \end{aligned} \tag{12}$$

Here we have used Eq. (3) underneath the integral and the performed the y integrals which are elementary.

To obtain the BCH formula we define

$$e^{xA}e^{xB} = e^{G(x)} = e^{xG_1 + x^2G_2 + x^3G_3 + ...} \,, \tag{13}$$

and consider

$$e^{-xB}e^{-xA}\frac{d}{dx}e^{xA}e^{xB} = e^{-G(x)}\frac{d}{dx}e^{G(x)} \,. \tag{14}$$

The l.h.s. of Eq. (14) is easily evaluated:

$$\begin{aligned} e^{-xB}e^{-xA}\frac{d}{dx}e^{xA}e^{xB} &= e^{-xB}Be^{xB} + e^{-xB}e^{-xA}Ae^{xA}e^{xB} \\ &= B + e^{-xB}Ae^{xB} \\ &= B + A + x[A, B] + \frac{x^2}{2}[B, [B, A]] + ... \end{aligned} \tag{15}$$

while

$$e^{-G(x)}\frac{d}{dx}e^{G(x)} = G' + \frac{1}{2}[G', G] + \frac{1}{3!}[[G', G], G] + ... \tag{16}$$
$$= G_1 + 2xG_2 + x^2\left(3G_3 - \frac{1}{2}[G_1, G_2]\right) + O(x^3)$$

where we have used

$$G' = G_1 + 2xG_2 + 3x^2G_3 + ... \tag{17}$$

and

$$\begin{aligned} [G', G] &= [G_1 + 2xG_2 + 3x^2G_3 + ..., xG_1 + x^2G_2 + x^3G_3 + ...] \\ &= x^2[G_1, G_2] + 2x^2[G_2, G_1] + O(x^3) \\ &= -x^2[G_1, G_2] + O(x^3) \end{aligned} \tag{18}$$

Note that $\frac{1}{3!}[[G', G], G]$ is $O(x^3)$ and so we can drop this term in Eq. (16). Equating Eq. (15) and Eq. (16) to $O(x^2)$ we obtain

$$\begin{aligned} G_1 &= A + B \\ G_2 &= \frac{1}{2}[A, B] \\ G_3 &= \frac{1}{12}\Big([A, [A, B]] + [B, [B, A]]\Big) \end{aligned} \tag{19}$$

Thus,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[B,[B,A]]+...}, \tag{20}$$

Higher order terms can be systematically, if tediously, computed by expanding both Eq. (15) and Eq. (16) to higher orders in x.