Cumulant Expansion for Quantum Dynamics

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Liouville form of time-evolution of a operator:

H-representation

$$\langle A(t)\rangle = \langle U^{\dagger}\cdot A_s \cdot U\rangle$$

ensemble average (if needed)

super operator $H^{\times}(\tau)\cdot A = [H(\tau), A] \equiv L(\tau)\cdot A$

$$= \left\langle \exp_{-}\left[\frac{-i}{\hbar}\int_0^t H^{\times}(\tau)\cdot d\tau\right]\right\rangle A_s$$

average

$$= \overline{\exp_{-}\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]}\cdot A_s$$

$\boxed{L\cdot A = [H, A]}$

Cumulant expansion:

$$\langle e^{B}\rangle = e^{\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n K_n}$$

$$\overline{\exp_{-}\left[\frac{-i}{\hbar}\int_0^t H^{\times}(\tau)\cdot d\tau\right]} = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot K_n^{\times}(t)\right]$$

that is, we want to write:

$$\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot K_n^{\times}(t) = \ln\left\{\overline{\exp_{-}\left[\frac{-i}{\hbar}\int_0^t H^{\times}(\tau)\, d\tau\right]}\right\}$$

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$$\therefore \text{RHS} = \ln\left\{ 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \cdot \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 \times \overline{\overline{H^\times(\tau_1)\cdot H^\times(\tau_2)\cdots H^\times(\tau_n)}} \right\}$$

inverse time ordering,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots + (-1)^{n-1}\cdot\frac{x^n}{n} + \cdots$$

$$= \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \cdot \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 \cdot \overline{H^\times(\tau_1)\cdot H^\times(\tau_2)\cdots H^\times(\tau_n)} \,,$$

← lowest order is $\left(\frac{i}{\hbar}\right)^1$

ensemble average first, then take power 2.

$$- \frac{1}{2}\times\left[\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \cdot \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 \cdot \overline{H^\times(\tau_1)\cdot H^\times(\tau_2)\cdots H^\times(\tau_n)}\right]^2$$

← lowest order is $\left(\frac{i}{\hbar}\right)^2$

$$+ \frac{1}{3}\cdot\left[\qquad \overline{\qquad} \qquad\right]^3$$

← lowest order is $\left(\frac{i}{\hbar}\right)^3$

$$+ \cdots \qquad + (-1)^{n-1}\cdot\frac{1}{n}\cdot\left[\,\overline{-}\,\right]^n + \cdots$$

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Compare by the order of $\left(\frac{i}{\hbar}\right)^n$, term by term. ③

$n=1$, LHS : $\frac{-i}{\hbar}\cdot K_1^{\times}(t)$

RHS : $\frac{-i}{\hbar}\cdot\int_0^t d\tau\cdot\overline{H^{\times}(\tau)}$

$n=2$, LHS : $\frac{1}{2}\cdot\left(\frac{-i}{\hbar}\right)^2\cdot K_2^{\times}(t)$

RHS : $\left(\frac{-i}{\hbar}\right)^2\cdot\int_0^t d\tau_2\int_0^{\tau_2} d\tau_1\cdot\overline{H^{\times}(\tau_1)\cdot H^{\times}(\tau_2)} - \frac{1}{2}\cdot\left[\frac{-i}{\hbar}\int_0^t d\tau\cdot\overline{H^{\times}(\tau)}\right]^2$

leads to TCFs

$n=3$, LHS $=\frac{1}{6}\cdot\left(\frac{-i}{\hbar}\right)^3\cdot K_3^{\times}(t)$

RHS : $\left(\frac{-i}{\hbar}\right)^3\cdot\int_0^t d\tau_3\int_0^{\tau_3} d\tau_2\int_0^{\tau_2} d\tau_1\cdot\overline{H^{\times}(\tau_1)\cdot H^{\times}(\tau_2)\cdot H^{\times}(\tau_3)}$

$-\frac{1}{2}\cdot\left(\frac{-i}{\hbar}\right)^3\cdot\left[\int_0^t d\tau\cdot\overline{H^{\times}(\tau)}\right]\left[\int_0^t d\tau_2\int_0^{\tau_2} d\tau_1\cdot\overline{H^{\times}(\tau_1)\cdot H^{\times}(\tau_2)}\right]$

$-\frac{1}{2}\left(\frac{-i}{\hbar}\right)^3\cdot\left[\int_0^t d\tau_2\int_0^{\tau_2} d\tau_1\cdot\overline{H^{\times}(\tau_1) H^{\times}(\tau_2)}\right]\left[\int_0^t d\tau\cdot\overline{H^{\times}(\tau)}\right]$

$+\frac{1}{3}\left(\frac{-i}{\hbar}\right)^3\cdot\left[\int_0^t d\tau\cdot\overline{H^{\times}(\tau)}\right]^3$

$$n=3,\ \text{RHS}:\ \left(\frac{i}{\hbar}\right)^3 \cdot \int_0^t d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot H^X(\tau_1)\cdot H^X(\tau_2)\cdot H^X(\tau_3)$$

$$-\frac{1}{2}\cdot\left[\frac{i}{\hbar}\cdot\int_0^t d\tau\cdot H^X(\tau)\right]\left[\left(\frac{i}{\hbar}\right)^2\cdot\int_0^t d\tau_2\int_0^{\tau_2} d\tau_1\cdot H^X(\tau_1)H^X(\tau_2)\right]$$

$$-\frac{1}{2}\cdot\left[\left(\frac{i}{\hbar}\right)^2\cdot\int_0^t d\tau_2\int_0^{\tau_2} d\tau_1\cdot H^X(\tau_1)\cdot H^X(\tau_2)\right]\left[\frac{i}{\hbar}\int_0^t d\tau\cdot H^X(\tau)\right]$$

$$+\frac{1}{3}\left(\frac{i}{\hbar}\right)^3\cdot\left[\int_0^t d\tau\cdot H^X(\tau)\right]^3 +$$

So we can obtain terms of $K_n^x(t)$.

$$K_1^x(t) = \int_0^t d\tau \cdot \overline{H^x(\tau)}$$

ensemble average

$$K_2^x(t) = 2\cdot \int_0^t d\tau_2 \cdot \int_0^{\tau_2} d\tau_1 \cdot \overline{H^x(\tau_1)\cdot H^x(\tau_2)} - \cancel{4}\left[\int_0^t d\tau \cdot \overline{H^x(\tau)}\right]^2$$

$$= 2\cdot \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \overline{H^x(\tau_1)\cdot H^x(\tau_2)} - \left[K_1^x(t)\right]^2$$

$$K_3^x(t) = \overset{=6}{3!}\cdot \int_0^t d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \overline{H^x(\tau_1)\cdot H^x(\tau_2)\cdot H^x(\tau_3)}$$

$$- 3\cdot K_1^x(t)\cdot\left[\frac{1}{2}K_2^x(t) + \frac{1}{2}\left[K_1^x(t)\right]^2\right]$$

$$- 3\cdot\left[\frac{1}{2}\cdot K_2^x(t) + \frac{1}{2}\left[K_1^x(t)\right]^2\right] K_1^x(t)$$

$$+ 2\cdot\left[K_1^x(t)\right]^3$$

$$= 6\cdot\int_0^t d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \overline{H^x(\tau_1)\cdot H^x(\tau_2)\cdot H^x(\tau_3)}$$

$$- \frac{3}{2} K_1^x(t)\cdot K_2^x(t) - \frac{3}{2} K_2^x(t)\cdot K_1^x(t) - \left[K_1^x(t)\right]^3$$

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Cumulents:

$C_1(x) = \langle x \rangle$

$C_2(x) = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle - C_1^2(x)$

$C_3(x) = \langle x^3 \rangle - 3 \cdot \langle x \rangle \langle x^2 \rangle + 2 \cdot \langle x \rangle^3 = \langle x^3 \rangle - 3 \cdot C_1(x) \cdot C_2(x) - \langle x \rangle^3$

$C_4(x) = \langle x^4 \rangle - 3 \langle x^2 \rangle^2 - 4 \langle x \rangle \langle x^3 \rangle + 12 \cdot \langle x \rangle^2 \langle x^2 \rangle$
$- 6 \langle x \rangle^4$

$3 C_1 \cdot C_2 = 3 \cdot \langle x \rangle \cdot (\langle x^2 \rangle - \langle x \rangle^2)$

$= 3 \cdot \langle x \rangle \langle x^2 \rangle - 3 \langle x \rangle^3$

$\therefore -3 \langle x \rangle \langle x^2 \rangle = -3 C_1 C_2 \, \cancel{+} - 3 \langle x \rangle^3$.

Quantum Master Equations

$$\because \langle A(t)\rangle = \left\langle \exp_{-}\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]\right\rangle A_s$$

cumulant after ensemble averages

$$= \exp\left[\sum_{n\geq 1}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \cdot \overline{K_n^X(t)}\right]\cdot A_s$$

actually $L(t)$

$$\therefore \frac{d}{dt}\langle A(t)\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot \overline{K_n^X(t)}\right]\cdot\left\{\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot\frac{d}{dt}\overline{K_n^X(t)}\right\}\cdot A_s$$

$$= \left\langle \exp_{-}\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]\right\rangle\cdot\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot\left[\frac{d}{dt}\overline{K_n^X(t)}\right]\cdot A_s$$

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note that $\frac{d}{dt}\overline{K_n^X(t)}\cdot A_s$ will generate various new operators in the S-representation, and

operator in S-represen

$$\left\langle \exp_{-}\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]\right\rangle\cdot O_s \equiv \langle O_s(t)\rangle$$

$\therefore$ we can have a system of equations for time dependent operators.

example: in TLS, use $a_1^\dagger a_1, a_2^\dagger a_2, a_1^\dagger a_2, a_2^\dagger a_1$, and we obtain 4 differential eq. describes $\rho_{11}(t)$, $\rho_{22}(t)$, $\rho_{12}(t)$ & $\rho_{21}(t)$

$\Rightarrow$ use Laplace transform to solve it!!

Example :

two-sites HSR model.

$$H = \varepsilon_1(t)\cdot a_1^\dagger a_1 - \varepsilon_2(t)\cdot a_2^\dagger a_2 + J(t)\cdot(a_1^\dagger a_2 + a_2^\dagger a_1)$$

$$= \varepsilon_0\cdot(a_1^\dagger a_1 - a_2^\dagger a_2) + J_0\cdot(a_1^\dagger a_2 + a_2^\dagger a_1) \quad) H_0$$

$$+ \delta\varepsilon_1(t)\cdot a_1^\dagger a_1 + \delta\varepsilon_2(t)\cdot a_2^\dagger a_2 + \delta J(t)\cdot(a_1^\dagger a_2 + a_2^\dagger a_1) \quad) V(t)$$

$$\equiv H_0 + V(t)$$

Stochastic process :

$$\langle \delta\varepsilon_n(t)\rangle = 0$$

$$\langle \delta\varepsilon_n(t_1)\cdot\delta\varepsilon_m(t_2)\rangle = \gamma_0\cdot\delta(t_1-t_2)\cdot\delta_{n,m}$$

$$\langle \delta J(t)\rangle = 0$$

$$\langle \delta J(t_1)\cdot\delta J(t_2)\rangle = \gamma_1\cdot\delta(t_1-t_2)$$

$$\therefore\quad K_1^\times(t) = \overline{\int_0^t d\tau\cdot[H_0^\times + V^\times(\tau)]}$$

$$= H_0^\times\cdot t + \int_0^t \overline{V^\times(\tau)}\cdot d\tau \longrightarrow \langle V^\times(\tau)\rangle = 0$$

$$= H_0^\times\cdot t$$

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$$\overline{K_2^X(t)} = 2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \overline{\left[H_0^X + V^X(\tau_1)\right]\left[H_0^X + V^X(\tau_2)\right]} - \left(H_0^X \cdot t\right)^2$$

$$= 2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \left[\cancel{\overline{H_0^X \cdot V^X(\tau_2)} + \overline{V^X(\tau_1)\cdot H_0^X}} + \overline{V^X(\tau_1)\, V^X(\tau_2)}\right]$$

1'

$$= 2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \Big\langle \left[\delta\varepsilon_1(\tau_1)\cdot a_1^\dagger a_1 + \delta\varepsilon_2(\overset{\tau_1}{\downarrow})\, a_2^\dagger a_2 + \delta J(\tau_1)\cdot(a_1^\dagger a_2 + a_2^\dagger a_1)\right]^X$$

ensemble average

$$\times \left[\delta\varepsilon_1(\tau_2)\, a_1^\dagger a_1 + \delta\varepsilon_2(\tau_2)\, a_2^\dagger a_2 + \delta J(\tau_2)\,(a_1^\dagger a_2 + a_2^\dagger a_1)\right]^X \Big\rangle$$

$$= [a_1^\dagger a_1]^X [a_1^\dagger a_1]^X \cdot \overbrace{\boxed{2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \langle \delta\varepsilon_1(\tau_1)\,\delta\varepsilon_1(\tau_2)\rangle}}^{=\gamma_0\cdot t} + [a_2^\dagger a_2]^X [a_2^\dagger a_2]^X \cdot 2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \langle \delta\varepsilon_2(\tau_1)\,\delta\varepsilon_2(\tau_2)\rangle$$

$$+ [a_1^\dagger a_2 + a_2^\dagger a_1]^X [a_1^\dagger a_2 + a_2^\dagger a_1]^X \cdot \overbrace{\boxed{2\cdot\int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \cdot \langle \delta J(\tau_1)\,\delta J(\tau_2)\rangle}}^{=\gamma_1\cdot t}$$

$$= \left\{[a_1^\dagger a_1]^X [a_1^\dagger a_1]^X + [a_2^\dagger a_2]^X [a_2^\dagger a_2]^X\right\}\cdot \gamma_0 \cdot t + [a_1^\dagger a_2 + a_2^\dagger a_1]^X [a_1^\dagger a_2 + a_2^\dagger a_1]^X \cdot \gamma_1 \cdot t$$

1′ Integrand

$$= \delta\xi_1(\tau_1)\,\delta\xi_1(\tau_2)\,[a_1^\dagger a_1]^\times [a_1^\dagger a_1]^\times + \cancel{\delta\xi_1(\tau_1)\,\delta\xi_2(\tau_2)\,[a_1^\dagger a_1]^\times [a_2^\dagger a_2]^\times} + \cancel{\delta\xi_1(\tau_1)\,\delta J(\tau_2)\,[a_1^\dagger a_1]^\times [a_1^\dagger a_2 + a_2^\dagger a_1]^\times}$$

$$+ \cancel{\delta\xi_2(\tau_1)\,\delta\xi_1(\tau_2)\,[a_2^\dagger a_2]^\times [a_1^\dagger a_1]^\times} + \delta\xi_2(\tau_1)\,\delta\xi_2(\tau_2)\,[a_2^\dagger a_2]^\times [a_2^\dagger a_2]^\times + \cancel{\delta\xi_2(\tau_1)\,\delta J(\tau_2)\,[a_2^\dagger a_2]^\times [a_1^\dagger a_2 + a_2^\dagger a_1]^\times}$$

$$+ \cancel{\delta J(\tau_1)\,\delta\xi_1(\tau_2)\,[a_1^\dagger a_2 + a_2^\dagger a_1]^\times [a_1^\dagger a_1]^\times} + \cancel{\delta J(\tau_1)\,\delta\xi_2(\tau_2)\,[a_1^\dagger a_2 + a_2^\dagger a_1]^\times [a_2^\dagger a_2]^\times} + \delta J(\tau_1)\,\delta J(\tau_2)\,[a_1^\dagger a_2 + a_2^\dagger a_1]^\times [a_1^\dagger a_2 + a_2^\dagger a_1]^\times$$

$$= \delta\xi_1(\tau_1)\,\delta\xi_1(\tau_2)\,[a_1^\dagger a_1]^\times [a_1^\dagger a_1]^\times + \delta\xi_2(\tau_1)\,\delta\xi_2(\tau_2)\,[a_2^\dagger a_2]^\times [a_2^\dagger a_2]^\times$$

$$+ \delta J(\tau_1)\,\delta J(\tau_2)\,[a_1^\dagger a_2 + a_2^\dagger a_1]^\times [a_1^\dagger a_2 + a_2^\dagger a_1]^\times$$

follow the procedure in page 5.

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and notice that.

now $\langle \exp_-\left[\frac{-i}{\hbar}\int_0^t L(\tau)\,d\tau\right]\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot\overline{K_n^x(t)}\right]\cdot$

$$= e^{\frac{-i}{\hbar}\cdot H_0^x\cdot t + \overline{K_2^x(t)}\cdot\frac{1}{2}\cdot\left(\frac{-i}{\hbar}\right)^2}$$

$\therefore\ \overline{\dot{A}(t)} = \frac{d}{dt}\langle \exp_-\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]\cdot A_s\rangle$

$$= \langle \exp_-\left[\frac{-i}{\hbar}\int_0^t L(\tau)\cdot d\tau\right]\cdot L(t)\cdot A_s\rangle$$

where $L(t) = \sum_{n=1}^{\infty}\frac{1}{n!}\cdot\left(\frac{-i}{\hbar}\right)^n\cdot\frac{d}{dt}\overline{K_n^x(t)}$

up to second order, all higher cumulants are zero.

$$= \frac{-i}{\hbar}H_0^x + \frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\Big\{\gamma_0\cdot\left[(a_1^\dagger a_1)^x(a_1^\dagger a_1)^x + (a_2^\dagger a_2)^x(a_2^\dagger a_2)^x\right] + \gamma_1\cdot(a_1^\dagger a_2 + a_2^\dagger a_1)^x(a_1^\dagger a_2 + a_2^\dagger a_1)^x\Big\}$$

$H_0 = \varepsilon_0\cdot(a_1^\dagger a_1 - a_2^\dagger a_2) + J_0\cdot(a_1^\dagger a_2 + a_2^\dagger a_1)$

now we are ready to study the time-evolution of various operators

Commutation rules

We want to know: $a_1^\dagger a_1,\ a_2^\dagger a_2,\ a_1^\dagger a_2,\ a_2^\dagger a_1$

and notice

$$[a_i^\dagger a_j, a_m^\dagger a_n] = a_i^\dagger a_j a_m^\dagger a_n - a_m^\dagger a_n a_i^\dagger a_j$$

$$= \delta_{mj}\, a_i^\dagger a_n - \delta_{n,i} \cdot a_m^\dagger a_j$$

∴ $[a_1^\dagger a_1, a_1^\dagger a_1] = 0$

$[a_1^\dagger a_1, a_2^\dagger a_2] = 0$

$[a_1^\dagger a_1, a_1^\dagger a_2] = a_1^\dagger a_2$

$[a_1^\dagger a_1, a_2^\dagger a_1] = -a_2^\dagger a_1$

$[a_2^\dagger a_2, a_2^\dagger a_2] = 0$

$[a_2^\dagger a_2, a_1^\dagger a_2] = -a_1^\dagger a_2$

$[a_2^\dagger a_2, a_2^\dagger a_1] = a_2^\dagger a_1$

$[a_1^\dagger a_2, a_1^\dagger a_2] = 0$

$[a_1^\dagger a_2, a_2^\dagger a_1] = a_1^\dagger a_1 - a_2^\dagger a_2$

$[a_2^\dagger a_1, a_2^\dagger a_1] = 0$

To simplify the notations, let's define.

$$O_1 \equiv a_1^\dagger a_1$$

$$O_2 \equiv a_2^\dagger a_2$$

$$O_3 \equiv a_1^\dagger a_2 + a_2^\dagger a_1$$

$$\therefore L = \frac{-i}{\hbar} H_0^\times + \frac{1}{2}\cdot\left(\frac{-i}{\hbar}\right)^2 \cdot \left\{ \gamma_0 \cdot \left[O_1^\times O_1^\times + O_2^\times O_2^\times \right] + \gamma_1 \cdot O_3^\times O_3^\times \right\}$$

$$= \frac{-i}{\hbar} \cdot \left[\varepsilon_0 \cdot (O_1^\times - O_2^\times) + J_0 \cdot O_3^\times \right] - \frac{1}{2\hbar^2} \cdot \left\{ \gamma_0 \cdot \left[O_1^\times O_1^\times + O_2^\times O_2^\times \right] + \gamma_1 \cdot O_3^\times O_3^\times \right\}$$

$\hbar = 1$

$$= -i \left[\varepsilon_0 \cdot (O_1^\times - O_2^\times) + J_0 \cdot O_3^\times \right] - \frac{1}{2} \left\{ \gamma_0 \cdot \left[O_1^\times O_1^\times + O_2^\times O_2^\times \right] + \gamma_1 \cdot O_3^\times O_3^\times \right\}$$

We are interested in the dynamics of ρ, $\rho_{ij} = a_i^\dagger a_j \quad (i,j) = 1, 2$

i.e. $\overset{=\rho_{11}}{a_1^\dagger a_1}, \overset{=\rho_{22}}{a_2^\dagger a_2}, \overset{=\rho_{12}}{a_1^\dagger a_2}, \overset{=\rho_{21}}{a_2^\dagger a_1}$

$\therefore$ We will perform L on $\rho_{11}, \rho_{22}, \rho_{12}, \rho_{21}$.

$$O_1^\times \cdot \rho_{12} = [a_1^\dagger a_1, a_1^\dagger a_2] = a_1^\dagger a_2 \equiv \rho_{12}$$

$$O_1^\times \cdot O_1^\times \rho_{12} = [a_1^\dagger a_1, a_1^\dagger a_2] = a_1^\dagger a_2 \equiv \rho_{12}$$

$$O_2^\times \rho_{12} = [a_2^\dagger a_2, a_1^\dagger a_2] = -a_1^\dagger a_2 \equiv -\rho_{12}$$

$$O_2^\times O_2^\times \rho_{12} = [a_2^\dagger a_2, -a_1^\dagger a_2] = a_1^\dagger a_2 \equiv \rho_{12}$$

$$O_3^\times \cdot \rho_{12} = [a_1^\dagger a_2 + a_2^\dagger a_1, a_1^\dagger a_2] = [a_2^\dagger a_1, a_1^\dagger a_2] = - \cdot (a_1^\dagger a_1 - a_2^\dagger a_2)$$

$$\equiv \rho_{22} - \rho_{11}$$

$$O_3^\times O_3^\times \rho_{12} = O_3^\times \cdot (\rho_{22} - \rho_{11}) = O_3^\times \rho_{22} - O_3^\times \rho_{11}$$

$$= \rho_{12} - \rho_{21} - (\rho_{21} - \rho_{12}) = 2 \cdot (\rho_{12} - \rho_{21})$$

$$O_1^\times \rho_{21} = [a_1^\dagger a_1, a_2^\dagger a_1] = -a_2^\dagger a_1$$

$$O_1^\times O_1^\times \rho_{21} = [a_1^\dagger a_1, -a_2^\dagger a_1] = a_2^\dagger a_1$$

$$O_2^\times \rho_{21} = [a_2^\dagger a_2, a_2^\dagger a_1] = a_2^\dagger a_1$$

$$O_2^\times O_2^\times \rho_{21} = a_2^\dagger a_1$$

$$O_3^\times \rho_{21} = [a_1^\dagger a_2 + a_2^\dagger a_1, a_2^\dagger a_1] = \rho_{11} - \rho_{22}$$

$$O_3^\times O_3^\times \rho_{21} = O_3^\times \cdot (\rho_{11} - \rho_{22}) = \rho_{21} - \rho_{12} - (\rho_{12} - \rho_{21})$$

$$= 2 \cdot (\rho_{21} - \rho_{12})$$

for $\rho_{11} \Rightarrow a_1^\dagger a_1$.

1'

$$L\cdot\rho_{11} = -i\cdot J_0\cdot O_3^\times\cdot\rho_{11} - \frac{1}{2}\cdot\gamma_1\cdot O_3^\times O_3^\times\cdot\rho_{11}$$

$$= -i\cdot J_0\cdot(\rho_{21}-\rho_{12}) - \cancel{\tfrac{1}{2}}\cdot\gamma_1\cdot(\rho_{11}-\rho_{22})\times\cancel{2}$$

for $\rho_{22} \Rightarrow a_2^\dagger a_2$

2'

$$L\cdot\rho_{22} = -i\cdot J_0\cdot O_3^\times\rho_{22} - \frac{1}{2}\gamma_1\cdot O_3^\times O_3^\times\cdot\rho_{22}$$

$$= i\cdot J_0\cdot(\rho_{12}-\rho_{21}) + \cancel{\tfrac{1}{2}}\gamma_1\cdot\cancel{2}\cdot(\rho_{11}-\rho_{22})$$

for $\rho_{12} \Rightarrow a_1^\dagger a_2$

back of P.7.

$$L\cdot\rho_{12} = -i\cdot\left[\varepsilon_0\cdot(O_1^\times - O_2^\times) + J_0\cdot O_3^\times\right]\cdot\rho_{12} - \frac{1}{2}\cdot\left\{\gamma_0\cdot\left[O_1^\times O_1^\times + O_2^\times O_2^\times\right] + \gamma_1 O_3^\times O_3^\times\right\}\cdot\rho_{12}$$

$$= -i\cdot\varepsilon_0\cdot 2\cdot\rho_{12} \mp i J_0\cdot(\rho_{22}-\rho_{11}) - \cancel{\tfrac{1}{2}}\cdot\gamma_0\cdot\cancel{2}\cdot\rho_{12} - \cancel{\tfrac{1}{2}}\cdot\gamma_1\cdot\cancel{2}\cdot(\rho_{12}-\rho_{21})$$

for $\rho_{21} \Rightarrow a_2^\dagger a_1$

$$L\cdot\rho_{21} = -i\cdot\varepsilon_0\cdot\left[O_1^\times\rho_{21} - O_2^\times\rho_{21}\right] \mp i\cdot J_0\cdot O_3^\times\rho_{21} - \frac{1}{2}\left\{\gamma_0\cdot\left[O_1^\times O_1^\times + O_2^\times O_2^\times\right] + \gamma_1 O_3^\times O_3^\times\right\}\cdot\rho_{21}$$

$$= +2\cdot i\cdot\varepsilon_0\cdot\rho_{21} \mp i\cdot J_0\cdot(\rho_{11}-\rho_{22}) - \gamma_0\cdot\rho_{21} - \gamma_1\cdot(\rho_{21}-\rho_{12})$$

(1) $O_3^\times \rho_{11} = O_3^\times \cdot a_1^\dagger a_1$ ← table @ back of p.8

$= [a_1^\dagger a_2 + a_2^\dagger a_1, a_1^\dagger a_1]$

$= -[a_1^\dagger a_1, a_1^\dagger a_2] - [a_1^\dagger a_1, a_2^\dagger a_1]$

$= -a_1^\dagger a_2 + a_2^\dagger a_1 = \underset{\rho_{21}}{a_2^\dagger a_1} \underset{-\rho_{12}}{- a_1^\dagger a_2} \equiv \rho_{21} - \rho_{12}$

$O_3^\times O_3^\times \cdot \rho_{11} = O_3^\times \cdot (a_2^\dagger a_1 - a_1^\dagger a_2)$

$= [a_1^\dagger a_2 + a_2^\dagger a_1, a_2^\dagger a_1 - a_1^\dagger a_2]$

$= [a_1^\dagger a_2, a_2^\dagger a_1] - \cancel{[a_1^\dagger a_2, a_1^\dagger a_2]} + \cancel{[a_2^\dagger a_1, a_2^\dagger a_1]} - [a_2^\dagger a_1, a_1^\dagger a_2]$

$= 2 \cdot [a_1^\dagger a_2, a_2^\dagger a_1] = 2 [a_1^\dagger a_1 - a_2^\dagger a_2] \equiv [\rho_{11} - \rho_{22}] \times 2$

(2) $O_3^\times \cdot \rho_{22} = [a_1^\dagger a_2 + a_2^\dagger a_1, a_2^\dagger a_2] = -[a_2^\dagger a_2, a_1^\dagger a_2] - [a_2^\dagger a_2, a_2^\dagger a_1]$

$= a_1^\dagger a_2 - a_2^\dagger a_1 = \rho_{12} - \rho_{21}$

$O_3^\times O_3^\times \rho_{22} = [a_1^\dagger a_2 + a_2^\dagger a_1, a_1^\dagger a_2 - a_2^\dagger a_1] = -[a_1^\dagger a_2, a_2^\dagger a_1] + [a_2^\dagger a_1, a_1^\dagger a_2]$

$= -2 \cdot [a_1^\dagger a_2, a_2^\dagger a_1] = -2 \cdot (a_1^\dagger a_1 - a_2^\dagger a_2)$

$= -2 \cdot (\rho_{11} - \rho_{22})$

∴, master-equations :

$$\dot{\rho}_{11}(t) = \langle a^{\dagger}\dot{a}_1(t)\rangle$$

$$= -i\cdot J_0\cdot\left[\rho_{21}(t)-\rho_{12}(t)\right] - \gamma_1\cdot\left[\rho_{11}(t)-\rho_{22}(t)\right]$$

$$\dot{\rho}_{22}(t) = -i\cdot J_0\cdot\left[\rho_{12}(t)-\rho_{21}(t)\right] + \gamma_1\cdot\left[\rho_{11}(t)-\rho_{22}(t)\right]$$

$$\dot{\rho}_{12}(t) = -2\cdot i\cdot\varepsilon_0\cdot\rho_{12}(t) \mp i\cdot J_0\cdot\left[\underset{\uparrow\,22}{\rho_{22}(t)}-\rho_{11}(t)\right] - \gamma_0\cdot\rho_{12}(t) - \gamma_1\cdot\left[\rho_{12}(t)-\rho_{21}(t)\right]$$

$$\dot{\rho}_{21}(t) = +2i\varepsilon_0\cdot\rho_{21}(t) \mp iJ_0\cdot\left[\rho_{11}(t)-\rho_{22}(t)\right] - \gamma_0\cdot\rho_{21}(t) - \gamma_1\cdot\left[\rho_{21}(t)-\rho_{12}(t)\right]$$

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The linear equations can be solved using Laplace transformation.

use : $\tilde{\rho}_{ij}(z) = \int_0^{\infty} e^{-z\cdot t}\cdot\rho_{ij}(t)\cdot dt \equiv \mathcal{L}\cdot\rho_{ij}$

and $\mathcal{L}\cdot\dot{\rho}_{ij}(t) = z\cdot\tilde{\rho}_{ij}(z) - \rho_{ij}(0)$

∴ $\cancel{z\cdot\tilde{\rho}_{11}(z) = ij}$

$$\boxed{\begin{aligned}
z\cdot\tilde{\rho}_{11} - \rho_{11}(0) &= -iJ_0\cdot\left[\tilde{\rho}_{21}-\rho_{12}\right] - \gamma_1\cdot\left[\tilde{\rho}_{11}-\tilde{\rho}_{22}\right]\\
z\cdot\tilde{\rho}_{22} - \rho_{22}(0) &= -iJ_0\cdot\left[\tilde{\rho}_{12}-\tilde{\rho}_{21}\right] + \gamma_1\cdot\left[\tilde{\rho}_{11}-\tilde{\rho}_{22}\right]\\
z\cdot\tilde{\rho}_{12} - \rho_{12}(0) &= -2\cdot i\cdot\varepsilon_0\cdot\tilde{\rho}_{12} \mp iJ_0\cdot\left[\tilde{\rho}_{22}-\tilde{\rho}_{11}\right] - \gamma_0\cdot\tilde{\rho}_{12} - \gamma_1\cdot\left[\tilde{\rho}_{12}-\tilde{\rho}_{21}\right]\\
z\cdot\tilde{\rho}_{21} - \rho_{21}(0) &= +2i\varepsilon_0\tilde{\rho}_{21} \mp iJ_0\cdot\left[\tilde{\rho}_{11}-\tilde{\rho}_{22}\right] - \gamma_0\cdot\tilde{\rho}_{21} - \gamma_1\cdot\left[\tilde{\rho}_{21}-\rho_{12}\right]
\end{aligned}}$$

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yccheng

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