Lecture 12

Properties of Quantum Angular Momentum

Study Goal of This Lecture

- Angular momentum operators
- Expectation values and measurements
- Heisenberg uncertainty principle

12.1 Review

We have discussed the eigenvalues and eigenfunctions of quantum rigid rotors. There, the quantization of the angular degrees of freedom, θ and ϕ, leads to two quantum numbers:

l: angular momentum quantum number
m: magnetic quantum number

The eigenfunctions, called spherical harmonics, are specified by the two quantum numbers.

$$\hat{L}^2 Y_l^m(\theta,\phi) = l(l+1)\hbar^2 Y_l^m(\theta,\phi),\ l = 0, 1, 2, \cdots , \tag{12.1}$$

$$\hat{L}_z Y_l^m(\theta,\phi) = m\hbar Y_l^m(\theta,\phi),\ m = \pm l, \pm(l-1), \cdots , 0\ . \tag{12.2}$$

We could also write this in Dirac notation:

$$|l,m\rangle = Y_l^m(\theta,\phi), \tag{12.3}$$

so

$$\hat{L}^2\,|l,m\rangle = l(l+1)\hbar^2\,|l,m\rangle\,, \tag{12.4}$$

$$\hat{L}_z\,|l,m\rangle = m\hbar\,|l,m\rangle\,. \tag{12.5}$$

Note that we have also defined:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \tag{12.6}$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \tag{12.7}$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \tag{12.8}$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x. \tag{12.9}$$

In this lecture, we will return to cover some basic properties of the angular momentum operators.

12.2 Commutators

Let's first examine the commutators. Recall the canonical quantum commutator:

$$[\hat{x},\hat{p}_x] = i\hbar,\;\; [\hat{x},\hat{p}_y] = 0. \tag{12.10}$$

We can evaluate

$$\begin{aligned}
\left[\hat{L}_x,\hat{L}_z\right] &= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) - (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) \\
&= \hat{y}\hat{p}_z\hat{z}\hat{p}_x - \hat{y}\hat{p}_z\hat{x}\hat{p}_z - \hat{z}\hat{p}_y\hat{z}\hat{p}_x + \hat{z}\hat{p}_y\hat{x}\hat{p}_z \\
&- \hat{z}\hat{p}_x\hat{y}\hat{p}_z + \hat{z}\hat{p}_x\hat{z}\hat{p}_y + \hat{x}\hat{p}_y\hat{y}\hat{p}_z - \hat{x}\hat{p}_z\hat{z}\hat{p}_y \\
&= \hat{y}\hat{p}_x[\hat{p}_z,\hat{z}] + \hat{x}\hat{p}_z[\hat{z},\hat{p}_z] \\
&= -i\hbar\hat{y}\hat{p}_x + i\hbar\hat{x}\hat{y} \\
&= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) = i\hbar\hat{L}_z.
\end{aligned} \tag{12.11}$$

Similarly,

$$\left[\hat{L}_y,\hat{L}_z\right] = i\hbar\hat{L}_x,\;\; \left[\hat{L}_z,\hat{L}_x\right] = i\hbar\hat{L}_y. \tag{12.12}$$

Recognize the "cyclic rule".

Now we clearly show $\left[\hat{L}_x,\hat{L}_y\right] \neq 0$, then how about $\left[\hat{L}^2,\hat{L}_x\right]$? We are ready to do the calculation:

$$
\begin{aligned}
\left[\hat{L}^2, \hat{L}_x\right] &= (\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)\hat{L}_x - \hat{L}_x\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \\
&= \cancel{\hat{L}_x^3} + \hat{L}_y\hat{L}_y\hat{L}_x + \hat{L}_z\hat{L}_z\hat{L}_x - \cancel{\hat{L}_x^3} - \hat{L}_x\hat{L}_y\hat{L}_y - \hat{L}_x\hat{L}_z\hat{L}_z \\
&= \hat{L}_y(\hat{L}_x\hat{L}_y - i\hbar\hat{L}_z) + \hat{L}_z(\hat{L}_x\hat{L}_z + i\hbar\hat{L}_y) - \hat{L}_x\hat{L}_y\hat{L}_y - \hat{L}_x\hat{L}_z\hat{L}_z \\
&= \hat{L}_y\hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_x\hat{L}_z + i\hbar\hat{L}_z\hat{L}_y - \hat{L}_x\hat{L}_y\hat{L}_y - \hat{L}_x\hat{L}_z\hat{L}_z \\
&= (\hat{L}_x\hat{L}_y - i\hbar\hat{L}_z)\hat{L}_y - i\hbar\hat{L}_y\hat{L}_z + (\hat{L}_x\hat{L}_z + i\hbar\hat{L}_y)\hat{L}_z + i\hbar\hat{L}_z\hat{L}_y - \hat{L}_x\hat{L}_y\hat{L}_y - \hat{L}_x\hat{L}_z\hat{L}_z \\
&= 0.
\end{aligned}
\tag{12.13}
$$

$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$

$\therefore \hat{L}_z\hat{L}_x = \hat{L}_x\hat{L}_y - i\hbar\hat{L}_z$

Similarly, one can find

$$\left[\hat{L}^2, \hat{L}_y\right] = 0, \ \left[\hat{L}^2, \hat{L}_z\right] = 0. \tag{12.14}$$

These prove what we described on the previous lecture.($\hat{L}^2$ and $\hat{L}_z$ can share the same eigenfunctions.)

12.2.1 Expectation Values

Now any properties related to angular momentum(spin? momentum? $\cdots$) could be evaluated by taking expectation values. Suppose:

$$|\Psi\rangle = \sum_{l,m} C_{l,m} |l, m\rangle. \tag{12.15}$$

Since set $\{|l, m\rangle\}$ is complete, this is always true. So given any operator regarding rotations:

$$\hat{A} = \hat{A}(\hat{L}_x, \hat{L}_y, \hat{L}_z), \tag{12.16}$$

$$\langle A\rangle = \langle\Psi|\hat{A}|\Psi\rangle = \sum_{l,m,l',m'} C^*_{l',m'} C_{l,m} \langle l', m'|\hat{A}|l, m\rangle. \tag{12.17}$$

Now recall that $\{|l, m\rangle\}$ are eigenstates of $\hat{L}^2$ and $\hat{L}_z$. So any operator $\hat{A} = (\hat{L}^2, \hat{L}_z)$ is easy to calculate.

E.g. $\hat{H} = \frac{\hat{L}^2}{2I}$.

$$\begin{aligned} \therefore \langle\Psi|\hat{H}|\Psi\rangle &= \sum_{l,m,l',m'} \frac{1}{2I} C^*_{l',m'} C_{l,m} \left\langle l', m' \middle| \hat{L}^2 \middle| l, m \right\rangle \\ &= \sum_{l,m,l',m'} \frac{1}{2I} C^*_{l',m'} C_{l,m} l(l+1)\hbar^2 \left\langle l', m' \middle| l, m \right\rangle \\ &= \frac{1}{2I} \sum_{l,m} |C_{l,m}|^2 l(l+1)\hbar^2. \end{aligned} \tag{12.18}$$

$|C_{l,m}|^2$ is the probability in state $|l, m\rangle$.
The problem is $\hat{A} = \hat{A}(\hat{L}_x, \hat{L}_y)$. Some are still simple:

E.g. $\hat{L}_x^2 + \hat{L}_y^2$.

$$\because \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \tag{12.19}$$

$$\therefore \langle l, m|(\hat{L}_x^2 + \hat{L}_y^2)|l, m\rangle = \langle l, m|(\hat{L}^2 - \hat{L}_z^2)|l, m\rangle = l(l+1)\hbar^2 - m^2\hbar^2. \tag{12.20}$$

This one has a simple geometrical interpretation:

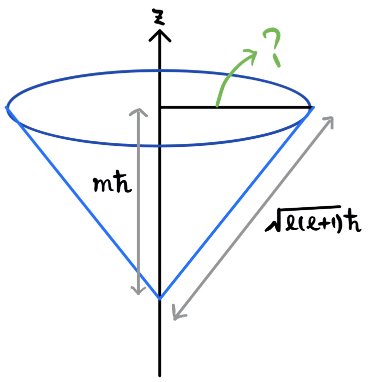

Figure 12.1: Geometric interpretation of $\hat{L}_x^2 + \hat{L}_y^2$

Finishing the simple one, but how about $\hat{L}_x$, $\hat{L}_y$? It's not so trivial. But look at the form $\hat{L}_x^2 + \hat{L}_y^2$ and remember what we did in treating harmonic oscillator based on the operator method, you should not be surprised by the following results:

Define

$$\begin{aligned} \hat{L_+} &= \hat{L}_x + i\hat{L}_y \\ \hat{L_-} &= \hat{L}_x - i\hat{L}_y \end{aligned} \Rightarrow \begin{aligned} \hat{L}_x &= \frac{1}{2}(\hat{L}_+ + \hat{L}_-) \\ \hat{L}_y &= \frac{-i}{2}(\hat{L}_+ - \hat{L}_-) \end{aligned} \tag{12.21}$$

$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hbar\hat{L}_\pm, \tag{12.22}$$

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z. \tag{12.23}$$

Where $\hat{L}_+$ and $\hat{L}_-$ is the ladder operator for angular momentum. We can derive further to obtain:

$$\hat{L}_+ |l, m\rangle = \sqrt{l(l+1) - m(m+1)}\hbar\, |l, m+1\rangle\,, \tag{12.24}$$

$$\hat{L}_- |l, m\rangle = \sqrt{l(l+1) - m(m-1)}\hbar\, |l, m-1\rangle\,, \tag{12.25}$$

and

$$\hat{L_+} |l, m = l\rangle = 0,\ \hat{L_-} |l, m = -l\rangle = 0. \tag{12.26}$$

These results can be dervied by the same procedure that we used to dervie the harmonic oscillator cases. However these are out of the scope in this class and it will not be given here.(See Levine's textbook)

The results above allow us to calculate any expectation values for functions of $\hat{L}_x, \hat{L}_y, \hat{L}_z$. For instance:

Try to check by yourself!

$$\langle l, m|\hat{L}_x|l, m\rangle = \langle l, m|\frac{1}{2}(\hat{L}_+ + \hat{L}_-)|l, m\rangle = 0, \tag{12.27}$$

$$\langle l, m|\hat{L}_x^2|l, m\rangle = \frac{1}{2}[l(l+1) - m^2]\hbar^2. \tag{12.28}$$

In undergraduate QM, we will not emphasize the ladder operator for angular momentum.

Can you guess the eigenvalues of $\hat{L}_x$ and $\hat{L}_y$?

12.3 Measurement

What if we can measure a single rotational state? we should obtain the following result:

$$\text{Measure } \hat{L}_z\text{, one gets } m\hbar,\ m = -l, \cdots, +l$$

$$\text{Measure } \hat{L}_x\text{, one gets } m\hbar,\ m = -l, \cdots, +l \text{ too!}$$

Since the "axis" is artificially labeled, the measurment results in three directions are identical(symmetry). But we know that $\left[\hat{L}_z, \hat{L}_x\right] \neq 0$, which means they do not share the same eigenbasis. So, an eigenfunction of $\hat{L}_z$ which yield single value when measuring $\hat{L}_z$ will be the superposition of the eigenfunctions of $\hat{L}_x$.

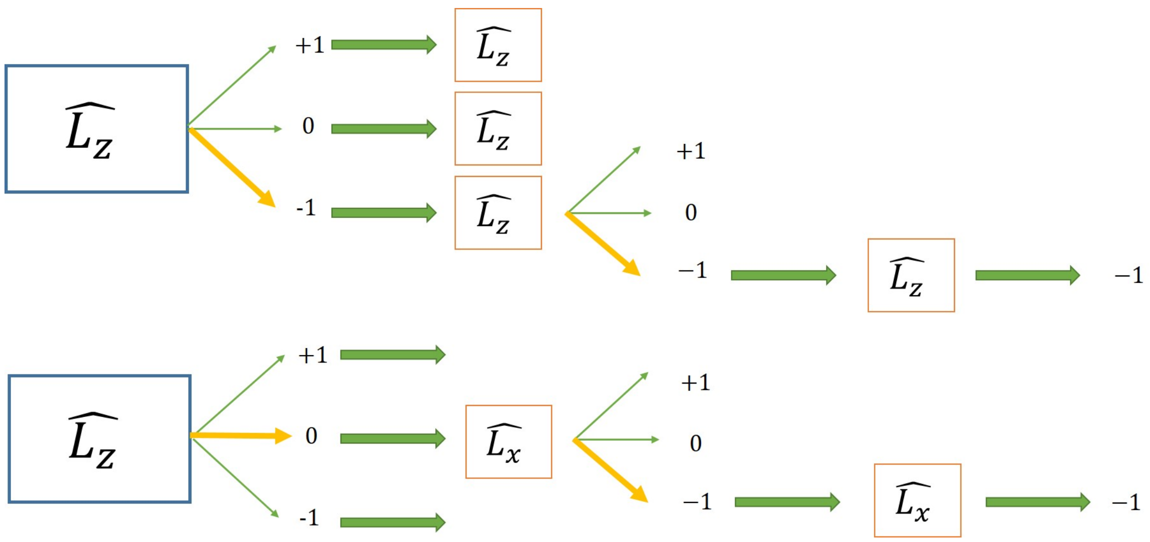

Figure 12.2: Measurement Tree.

Let's consider the consecutive measurement below:

For the case of $l = 1$, the possible measured values are $m = +1, 0, -1$. If we prepared an initial state of equal superposition of $m = +1, 0, -1$ states, then we can represent the measurement as a tree:(next page)

The boxes represent measurements and the yellow arrows represent the outcomes. From above we find that measurement of a different orthogonal component will "scramble" the states. This is obvious in the below one measurement in Fig.12.2. After the first $\hat{L}_z$ measurement, the state goes to the $m = 0$ state in $\hat{L}_z$ basis. And for the next $\hat{L}_x$ measurement, the states change to the $m = -1$ states in $\hat{L}_x$ basis. So we can re-state the Heisenberg uncertainty principle:

Uncertainty occurs because non-commuting measurements interfere with each other.

> If two obeservables do not commute, then an measurement of $\hat{A}$ will scramble the wavefunction to make the following measurement on $\hat{B}$ probabolistic. i.e. not exactly determined.

Or, one can not simultaneously determine $\hat{A}$ and $\hat{B}$ if $\left[\hat{A}, \hat{B}\right] \neq 0$. In fact, one can prove that

$$\langle(\Delta A)\rangle^2 \langle(\Delta B)\rangle^2 \geq \frac{1}{4} | \left\langle \left[\hat{A}, \hat{B}\right] \right\rangle |^2. \tag{12.29}$$

***Variance of $\hat{L}_x$ and $\hat{L}_y$ and the Uncertainty**

We can calculated, from physical intuitions, that given a eigenstate of $\hat{L}_z$, i.e. $|l,m\rangle$, then isotropic expectation values in x, y, z are, by symmetry

$$\langle l,m|\hat{L}_x|l,m\rangle = \langle l,m|\hat{L}_y|l,m\rangle = 0, \tag{12.30}$$

also because $\langle l,m|\hat{L}_x^2|l,m\rangle = \langle l,m|\hat{L}_y^2|l,m\rangle$, we can calculate using

$$\langle l,m|\hat{L}^2|l,m\rangle = \langle l,m|\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2|l,m\rangle\,, \tag{12.31}$$

$$\therefore\ \langle l,m|\hat{L}_x^2 + \hat{L}_y^2|l,m\rangle = 2\,\langle l,m|\hat{L}_x^2|l,m\rangle = \langle l,m|\hat{L}^2 - \hat{L}_z^2|l,m\rangle\,. \tag{12.32}$$

So

$$\langle l,m|\hat{L}_x^2|l,m\rangle = \langle l,m|\hat{L}_y^2|l,m\rangle = \frac{1}{2}\{l(l+1)\hbar^2 - m^2\hbar^2\}. \tag{12.33}$$

Therefore, uncertainty Δl_x and Δl_y for the $|l,m\rangle$ state is

$$\Delta l_x = \Delta l_y = \sqrt{\frac{1}{2}[l(l+1) - m^2]}\cdot\hbar. \tag{12.34}$$

12.4 Summary on Quantum Angular Momentum

Angular momentum is a vector, so one must specify its "magnitude" and "direction". In quantum mechanics, the magnitude is quantized for stationary states:

$$|\vec{L}| = \sqrt{l(l+1)}\hbar,\ \ l = 0, 1, 2, \cdots\,. \tag{12.35}$$

Since "magnitude" is determined, the direction can not be fully specified, otherwise the uncertainty principle will be violated. This is reflected in the fact that

$$\left[\hat{L}_x, \hat{L}_y\right] \neq 0,\ \left[\hat{L}_y, \hat{L}_z\right] \neq 0,\ \left[\hat{L}_z, \hat{L}_x\right] \neq 0. \tag{12.36}$$

But one direction can be specified because

$$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}^2, \hat{L}_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0. \tag{12.37}$$

By convention, we choose to specify $\hat{L}_y$, i.e. the z-component of $\vec{L}$:

$$l_z = m\hbar,\ \ m = -l, \cdots, 0, \cdots, +l. \tag{12.38}$$

Since z-component is determined, the l_x and l_y components are fully undetermined. This is best described by the vector diagram.

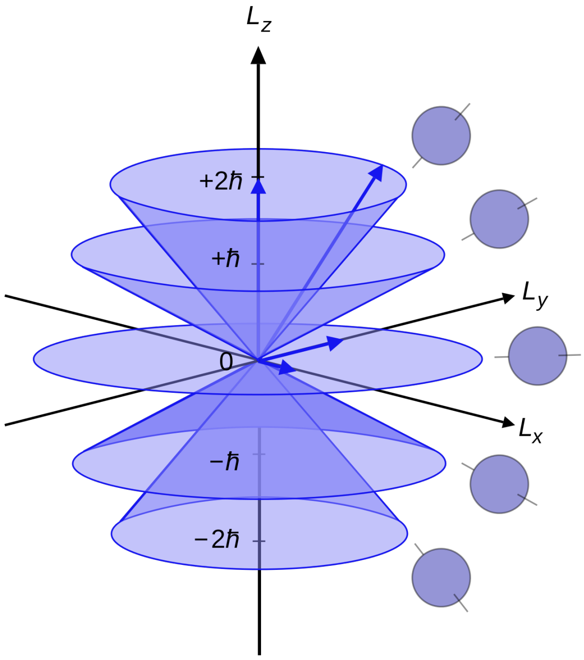

Figure 12.3: Vector diagram.

12.4.1 Measurement are Projections

$$\text{Measure } \hat{L}_z \to \text{project onto } \hat{z}.$$
$$\text{Measure } \hat{L}_x \to \text{project onto } \hat{x}.$$

Each measurement yields an eigenvalue and the state is changed to the corresponding eigenstate. So if one starts with $|l = 2, m = \pm 2\rangle \equiv |2, 2\rangle$.

- Then measuring of $\hat{L}_z$ yields $2\hbar$, the state remains the same (eigenstate, stationary). It Will not be affected by the projection. This is the same regardless of quantum or classical.
- How about measuring $\hat{L}_x$? Since $\left[\hat{L}_z, \hat{L}_x\right] \neq 0$, the projection could be anything between $-\sqrt{6-2^2}\hbar$ and $\sqrt{6-2^2}\hbar$

Comparison:

- Classical: A contineous range and values, the state remains at $|2, 2\rangle$ after measurement (no matter it is $\hat{L}_x$, $\hat{L}_y$ or $\hat{L}_z$).

- Quantum: Only could yield $-2\hbar, -\hbar, 0, \hbar, 2\hbar$. The state is changed to the correponding eigenstate of $\hat{L}_x$ after measurement. (Only being unchanged to the measurement of $\hat{L}_z$.)