Lecture 4

Quantum Measurement

Study Goal of This Lecture

- Observables are Hermitian operators
- Single measurement event
- Expectation values – averages
- Outcome of consecutive measurement of non-commutable operator

4.1 Measurement

With those useful results in linear algebra, we are ready to consider another important piece of quantum mechanics before we move on to treat real physical systems. The measurement, i.e. description of an experiment that can be carried out to obtain information about the system, is considered to be the most notorious part of quantum mechanics. As a newbee in QM, one is recommended to not think too much about the "interpretation" or "philosophical aspects" of quantum measurement, instead, take it that this is a "pratical procedure", and it works. Now we may officially introduce the following statement that was considered to be the most intriguing/controversial postulate in quantum mechanics:

> A measurement always cause the system jump into an eigenstate of the *dynamical variable* that is being measured. by *Paul A.M. Dirac*

Dynamical variable is the observable defined by the measuring apparatus.

as such, the only possible outcomes of an observable in a single experiment on a single system are the eigenvalues of the operator corresponding to that observable.

Why?

→ only eigenstate is stationary and stable enough.

Remark: Sometimes we also say "jumping into an eigenstate" as "collapse of wavefunction". Please note the above statement is for "a single measurement event!"

Therefore, given $\hat{A}\phi_n = a_n\phi_n$, then for a single measurement event yield:

$$\psi \xrightarrow[measurement]{\hat{A}} \begin{array}{cc} \textbf{Eigenfunctions} & \textbf{Eigenvalues(observed)} \\ \phi_1 & a_1 = \int \phi_1(\vec{r})\hat{A}\phi_1(\vec{r})d\tau \\ \phi_2 & a_2 \\ \phi_3 & a_3 \\ \vdots & \vdots \\ \phi_n & a_n \end{array}$$

The "collapse of wavefunction" is actually induced by the macroscopic measurement apparatus, although the details are hard to derive. This is the source of Einstein's "God does not play dice" quote, yet this does not mean QM is stochastic (not determinitisc). → there is a random outcome because of the "lack of information" in the macroscopic apparatus!

4.1.1 Measurement of a Physical Observable

Let's go back to measurement itself!

In general ψ is not an eigenfunction of $\hat{A}$. However, ψ can always be written as the linear combination of the eigenfunction of $\hat{A}$, i.e.

An important postulate in quantum mechanics. Actually, it is the key of QM!

$$\psi = \sum_n C_n\phi_n. \tag{4.1}$$

In linear algebra, we say that all possible eigenstate of $\hat{A}$, $\{\phi_n\}$, form a complete set, given that $\{\phi_n\}$ are orthornormal.

$$\int \phi_n^*\phi_m d\tau = \delta_{nm}. \tag{4.2}$$

We require from the normalization condition that

$$\sum_n |C_n|^2 = 1. \tag{4.3}$$

Proof.

$$1 = \int \psi^* \psi d\tau = \int (\sum_n C_n \phi_n)^* (\sum_m C_m \phi_m) d\tau$$

$$= \sum_{n,m} C_n^* C_m \int \phi_n^* \phi_m d\tau = \sum_{n,m} C_n^* C_m \delta_{nm} = \sum_n |C_n|^2 = 1$$

□

With the above property, if ψ is known, we can calculated C_n from multiply ϕ_m^* on both side of $\psi = \sum_n C_n \phi_n$:

$$\psi \phi_m^* = \sum_n C_n \phi_n \phi_m^*, \tag{4.4}$$

integrate both side:

$$\int \phi_m^* \psi d\tau = \int \sum_n C_n \phi_m^* \phi_n d\tau, \tag{4.5}$$

applying Equ(4.2):

$$\int \phi_m^* \psi d\tau = C_n. \tag{4.6}$$

C_n can also be a complex number.

This is also how we rewrite ψ as the superposition of $\{\phi_n\}$.

Normally, an "ensemble" measurement is carried out. In lecture 3, we have introduced the expectation value of operator $\hat{A}$

$$\langle a \rangle = \int \psi^* \hat{A} \psi d\tau, \tag{4.7}$$

we expand the ψ in the eigenbasis of $\hat{A}$

$$\langle a \rangle = \int \psi^* \hat{A} \psi d\tau = \int (\sum_n C_n^* \phi_n^*) \hat{A} (\sum_m C_m \phi_m) d\tau \tag{4.8}$$

$$= \sum_{n,m} C_n^* C_m \int \phi_n^* \hat{A} \phi_m d\tau$$

$$= \sum_{n,m} C_n^* C_m a_m \int \phi_n^* \phi_m d\tau$$

$$= \sum_n |C_n|^2 a_n. \tag{4.9}$$

therefore we interpret $|C_n|^2$ as the probability of jumping into state ϕ_n, and a_n is that particular outcome. In other word, $|C_n|^2$ is the probability of finding the system on state ϕ_n. $\Rightarrow$so expectation value $= \sum_n |C_n|^2 a_n$.
The more detail interpretation is that: for each single measurement, there is $|C_n|^2$ probability of obtaining a_n. Therefore, after large number times of measurements, we observe an average value $\langle a \rangle$.

For continuous observable, e.g. $\hat{x}$

$$\begin{aligned}\langle x \rangle &= \int \psi^* \hat{x} \psi d\tau = \int \psi^* x \psi d\tau \\ &= \int x |\psi|^2 d\tau.\end{aligned}$$

$\therefore$ $|\psi|^2$ is the probability density for the positive variable.
This is the microscopic view of measurement.

4.1.2 Uncertainty of Measurement

Now we can vigorously define the Δx and Δp in the uncertainty principle. Since quantum mechanical measurements on a single systme do not yield a dfinitive value, we very often want to also know the spread around the mean. i.e. the uncertainty of the measurement. This is measured by the variance $\Delta \hat{A}^2$.

$$\begin{aligned}\langle \Delta \hat{A}^2 \rangle &= \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle (\hat{A}^2 - \hat{A}\langle \hat{A} \rangle - \langle \hat{A} \rangle \hat{A} + \langle \hat{A} \rangle^2) \rangle \\ &= \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2.\end{aligned} \tag{4.10}$$

note that $\langle \hat{A} \rangle = \sum_n |C_n|^2 a_n$, $\langle \hat{A}^2 \rangle = \sum_n |C_n|^2 a_n^2$.

Proof.

$$\begin{aligned}\langle \hat{A}^2 \rangle &= \int \psi^* \hat{A}\hat{A} \psi d\tau \\ &= \int (\sum_n C_n^* \phi_n^*) \hat{A}\hat{A} (\sum_m C_m \phi_m) d\tau \\ &= \sum_{n,m} C_n^* C_m a_m^2 \cdot \int \phi_m^* \phi_n d\tau \\ &= \sum_n |C_n|^2 a_n^2.\end{aligned}$$

□

Now we carried all the basic rules in calculating quantum mechanical observables if ψ is given. We will apply what we learn so far to some simple model(e.g. particle in a box and hormonic oscillator) in the following lectures.

4.2 Consecutive Measurement of Non-commutable Operators

In previous lecture, we introduce the concept of commutator of two operators and show that if two operators commute, then they can share the same eigenfunction. After introducing the quantum measurement here, we might ask: **What would happen if we make a consecutive measurement of two commutable operators and non-commutable operator?**

Actually, it is the application of simultaneous diagnalization in linear algebra.

We look at the former case first. We have already known that the two commutable operators can share the same eigenfunctions, so if we prepare the system at initial state ψ and then do a consecutive measurement of operator $\hat{A}$ and $\hat{B}$ where $\left[\hat{A}, \hat{B}\right] = 0$.
Measure $\hat{A}$ firstly, we consider

$$\hat{A}\psi = \hat{A}\sum_n C_{a,n}\phi_{a,n}, \tag{4.11}$$

the probability of obtaining a_n value is $|C_{a,n}|^2$, assume the system becomes in state $\phi_{a,n}$ after measurement of $\hat{A}$.

$$\hat{A}\phi_{a,n} = a_n\phi_{a,n}. \tag{4.12}$$

Then we measure $\hat{B}$, since $\left[\hat{A}, \hat{B}\right] = 0$, the measurement of $\hat{B}$ may make the eigenfunction changes, but it is still in the eigenbasis of eigenvalue a_n, so we obtain

$$\hat{B}\phi_{a,n} = \sum_m C_{b,m,a,n}\phi_{b,m,a,n}. \tag{4.13}$$

The probability of obtaining a_n value is $|C_{b,m,a,n}|^2$, and we assume the system becomes in state $\phi_{b,m,a,n}$.

$$\hat{B}\phi_{b,m,a,n} = b_m\phi_{b,m,a,n}, \tag{4.14}$$

$\phi_{b,m,a,n}$ stands for the simultaneous eigenfunction of $\hat{A}$ and $\hat{B}$ at a_n and b_m eigenvalues respectively. The astonishing result is that, for later measurement of $\hat{A}$ and $\hat{B}$, we would always obtain the eigenvalue a_n and b_m

So we make a simple conclusion here: For the consecutive measurement of two commuatable operators $\hat{A}$ and $\hat{B}$, the system will end up in the simultaneous eigenstate of two operators, and for any afterward measurement of $\hat{A}$ and $\hat{B}$, the outcome will be the same. It also implies that there is no uncertainty for these two opeators.

Next, we consider the case of consecutive measurement of non-commuatable operators, where two operators can't share the same eigenfunction. For measuring of $\hat{A}$, we obtain the result as above:

$$\hat{A}\phi_{a,n} = a_n \phi_{a,n}. \tag{4.15}$$

then for the next measurement $\hat{B}$,

$$\hat{B}\phi_{a,n} = \hat{B}\sum_m C_{b,m}\phi_{b,m}. \tag{4.16}$$

Note that since $\left[\hat{A}, \hat{B}\right] \neq 0$, they won't share the same eigenfunction in general, so we can'y write the Eqn (4.17) as Eqn (4.14). Assume the system becomes in state $\phi_{b,m}$ after measurement, we now measure the observable $\hat{A}$ again, and find that

$$\hat{A}\phi_{b,m} = \hat{A}\sum_n C_{a,n}\phi_{a,n}. \tag{4.17}$$

This time, there is no gaurantee the outcome will be a_n as first measurement. So the system might become in state $\phi_{a,n'}$ and we obtaining the eigenvalue $a_{n'}$.

We summerize the discover here: For the consecutive measurement of two non-commutable operators. Since operators do not share the same eigenstate, when one operator operates on the eigenstate of the other one, the eigenstate will be "interfered". So when we do the same measurement after measuring the non-commutable operator, the system will not give us the same result as last time. This is the cause of uncertainty principle. We will give a concrete example when we mention the angular momentum operators.