Physics 116C

Singular Fourier transforms and the Integral Representation of the Dirac Delta Function

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I. INTRODUCTION

You will recall that Fourier transform, $g(k)$, of a function $f(x)$ is defined by

$$g(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx \,, \tag{1}$$

and that there is a very similar relation, the inverse Fourier transform,[1] transforming $g(k)$ back to $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{-ikx} \, dk \,, \tag{2}$$

One can show that, for the Fourier transform to converge as the limits of integration tend to $\pm\infty$, we must have $f(x) \to 0$ as $|x| \to \infty$. In addition, let's consider the $g_1(k)$, the Fourier transform of the derivative $f'(x)$, and call this $g_1(k)$, i.e.

$$g_1(k) = \int_{-\infty}^{\infty} f'(x) e^{ikx} \, dx \,, \tag{3}$$

Integrating by parts gives

$$g_1(k) = \Big[f(x) e^{ikx} \Big]_{-\infty}^{\infty} - (ik) \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx \,, \tag{4}$$

which shows that the standard relation between $g_1(k)$ and $g(k)$,

$$g_1(k) = -ikg(k) \,, \tag{5}$$

also only holds if $f(x)$ vanishes for $|x| \to \infty$. Hence *standard* Fourier transforms only apply to functions which vanish at infinity.

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[1] Sometimes the Fourier transform and its inverse are made equivalent by incorporating at factor of $1/\sqrt{2\pi}$ into the definition of $g(k)$ namely

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx \,,$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{-ikx} \, dk \,.$$

Nonetheless, Fourier transforms are so useful that it is desirable to apply them to some functions which do not satisfy this condition. These transforms are known as "*singular* Fourier transforms" and will need some form of "*regularization*" to make the integrals converge.

NOTE: for this course, the important sections are I, II, III, and V.

II. A SINGULAR FOURIER TRANSFORM INVOLVING A DELTA FUNCTION

As an example consider $f(x) = 1$. In order that the Fourier transform $g(k)$ exists, we regularize the integral by putting in the "convergence factor" $e^{-\epsilon|x|}$ where ϵ is small and positive. Eventually we will let ϵ tend to zero.

Hence we determine the Fourier transform of

$$f_\epsilon(x) = e^{-\epsilon|x|}, \tag{6}$$

which is

$$g_\epsilon(k) = \int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|}\, dx\,.$$

We separate the integral into the negative-x region and the positive-x region to find

$$g_\epsilon(k) = \int_{-\infty}^{0} e^{ikx} e^{\epsilon x}\, dx + \int_{0}^{\infty} e^{ikx} e^{-\epsilon x}\, dx = \frac{1}{ik+\epsilon} + \frac{1}{-ik+\epsilon} = \frac{2\epsilon}{\epsilon^2 + k^2}\,. \tag{7}$$

For $\epsilon \to 0$ $g_\epsilon(k)$, becomes a narrow high peak, the area under which is

$$\int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + k^2}\, dk = 2\left[\tan^{-1}(k/\epsilon)\right]_{-\infty}^{\infty} = 2\pi.$$

It is therefore convenient to define a quantity $\delta_\epsilon(k)$ by

$$\boxed{\delta_\epsilon(k) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + k^2}\,,} \tag{8}$$

which has unit area under it.

We now consider the limit $\epsilon \to 0^+$, for which $\delta_\epsilon(k)$ is a representation of what is known as the Dirac delta function $\delta(k)$. This is an "infinitely high, infinitely narrow" peak with unit area under it. It is defined by the two relations

$$\delta(x) = 0, \quad (x \neq 0), \tag{9}$$

$$\int \delta(x)\, dx = 1, \quad \text{(if region of integration includes } x = 0). \tag{10}$$

From these, it is straightforward to prove the following results:

$$\int \delta(x-a) f(x)\, dx \;=\; f(a), \tag{11}$$

$$\delta(cx) \;=\; \frac{\delta(x)}{|c|}\,, \tag{12}$$

where the region of integration in Eq. (11) includes $x = a$. You should have seen Eqs. (11) and (12) before. If you are unfamiliar with them, you should take the trouble to derive them.

From Eqs. (7) and (8) we have

$$\int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|}\, dx = 2\pi \delta_\epsilon(k). \tag{13}$$

This is a Fourier transform, for which I use the following notation:

$$\boxed{e^{-\epsilon|x|} \;\xrightarrow{FT}\; 2\pi\delta_\epsilon(k)\,.} \tag{14}$$

The integral in Eq. (13) is well defined because the $e^{-\epsilon|x|}$ factor ensures convergence.

Equations like Eq. (13) are generally used in situations when they are multiplied on both sides by a smooth function of k, $u(k)$ say, and integrated, *i.e.*

$$\int_{-\infty}^{\infty} u(k) \left[\int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|}\, dx \right] dk = 2\pi \int u(k) \delta_\epsilon(k)\, dk. \tag{15}$$

It turns out that this equation is well behaved if ϵ is set to zero on the LHS (and the limit $\epsilon \to 0$ is taken on the RHS). This gives

$$\begin{aligned} \int_{-\infty}^{\infty} u(k) \left[\int_{-\infty}^{\infty} e^{ikx}\, dx \right] dk \;&=\; 2\pi \int_{-\infty}^{\infty} u(k)\delta(k)\, dk \\ &=\; 2\pi\, u(0)\,, \end{aligned} \tag{16}$$

(which is known as Fourier's integral). We used Eq. (11) to get the last expression. One is often tempted to set ϵ to zero also in Eq. (13) (*i.e. without* multiplying it by a smooth function and integrating), in which case we write

$$\boxed{\int_{-\infty}^{\infty} e^{ikx}\, dx = 2\pi\delta(k)\,.} \tag{17}$$

However, as it stands, Eq. (17) *does not make sense* because the integral does not exist. We therefore have to understand Eq. (17) in one of the following two senses:

- As a stand-alone equation, in which case it has to be regularized by the convergence factor $e^{-\epsilon|x|}$, so Eq. (17) really means Eq. (13) for ϵ *tending* to zero (but not strictly zero).

- Multiplied by a smooth function $u(k)$ and integrated over k as in Eq. (15), in which case the convergence factor is unnecessary. Equation (17) is then really a shorthand for Eq. (16). This is normally the sense in which we understand Eq. (17).

Since Eq. (17) is a Fourier transform, we can write it as

$$\boxed{1 \xrightarrow{FT} 2\pi\delta(k)\,,} \tag{18}$$

which should be compared with Eq. (14). The inverse transform of the delta function then gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(k)e^{-ikx}\,dk = 1\,,$$

as required.

III. APPLICATIONS OF THE INTEGRAL REPRESENTATION OF THE DELTA FUNCTION

In this section we give some applications of the integral representation of the delta function, Eq. (17).

A. Convolution Theorem

In a previous class, you have already met the convolution theorem, that is, if

$$F(x) = \int_{-\infty}^{\infty} f(y)f(x-y)\,dy, \tag{19}$$

which is the convolution of f with itself, then $G(k)$, the Fourier transform of $F(x)$, is simply related to $g(k)$, the Fourier transform of $f(x)$, by

$$G(k) = g(k)^2\,. \tag{20}$$

We will now give a simple alternative derivation of this result using the integral representation of the delta function, and then use this method to obtain a *generalized* convolution theorem. We can write Eq. (19) as

$$F(x) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, f(x_1)f(x_2)\delta(x_1+x_2-x). \tag{21}$$

Using Eq. (17) we have

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dk\, f(x_1)f(x_2)e^{ik(x_1+x_2-x)}. \tag{22}$$

The integrals over x_1 and x_2 are now independent of each other and can be carried out, with the result that

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{ikt} \, dt \right]^2 e^{-ikx} \, dk, = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)^2 e^{-ikx} \, dk, \tag{23}$$

which shows that $F(x)$ is the inverse transform of $g(k)^2$, *i.e.* that $g(k)^2$ is the Fourier transform of $F(x)$. Hence we have obtained Eq. (20). The derivation immediately generalizes to the case of two different functions $f_1(x)$ and $f_2(x)$, with Fourier transforms $g_1(k)$ and $g_2(k)$, namely, the Fourier transform of the convolution

$$F(x) = \int_{-\infty}^{\infty} f_1(y) \, f_2(x-y) \, dy \, ,$$

is given by $G(k)$ where

$$G(k) = g_1(k) g_2(k) \, . \tag{24}$$

We can now generalize this result to the case where the convolution, rather than involving two variables as in Eq. (21), involves n variables, $x_1, x_2, \cdots, x_n$, but with the same constraint that the sum must equal some prescribed value x, *i.e.*

$$\boxed{F(x) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \, f(x_1) \cdots f(x_n) \, \delta(x_1 + \cdots + x_n - x).} \tag{25}$$

Using the integral representation of the delta function, as before, the integrals over the x_i decouple, and we find

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{ikt} \, dt \right]^n e^{-ikx} \, dk, = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)^n e^{-ikx} \, dk, \tag{26}$$

which shows that the Fourier transform of $F(x)$ is

$$\boxed{G(k) = g(k)^n,} \tag{27}$$

a remarkably simple result. Equation (27) is the desired generalization of the convolution theorem, Eq. (20), to n variables. We shall use this result in class to solve a problem in statistics.

Note: With the alternative definition of Fourier transforms given in footnote 1, which puts in factors of $\sqrt{2\pi}$ to make the transform and the inverse transform symmetric with respect to each other, the convolution theorem for n variables is

$$\sqrt{2\pi} G(k) = \left[\sqrt{2\pi} g(k)^n \right] . \tag{28}$$

B. Parseval's Theorem

Related to the convolution theorem is another useful theorem associated with the name of Parseval. (You may recall that there is a Parseval's theorem for Fourier series, which is actually closely related.)

Using the Fourier transform

$$g(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx \tag{29}$$

we have

$$\int_{-\infty}^{\infty} g(k) g^{\star}(k) \, dk = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dk \, f(x_1) f^{\star}(x_2) e^{ik(x_1 - x_2)}. \tag{30}$$

Doing the integral over k using Eq. (17) gives $2\pi\delta(x_1 - x_2)$, so

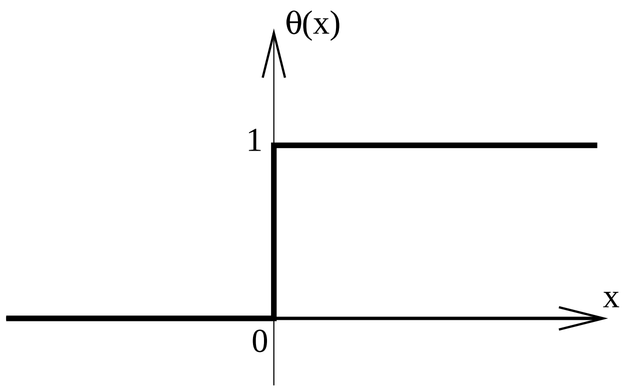
$$\begin{aligned} \int_{-\infty}^{\infty} |g(k)|^2 \, dk &= 2\pi \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, f(x_1) f^{\star}(x_2) \delta(x_1 - x_2) \\ &= \boxed{2\pi \int_{-\infty}^{\infty} |f(x)|^2 \, dx,} \end{aligned} \tag{31}$$

which is Parseval's theorem.

Note: With the alternative definition of Fourier transforms, the factor of 2π in Eq. (31) is missing, so there is complete symmetry between the two sides.

IV. EXAMPLES OF SINGULAR FOURIER TRANSFORMS INVOLVING A STEP FUNCTION

It is also interesting to consider singular Fourier transforms of functions involving the (Heaviside) step function



$$\theta(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases} ,$$

which is denoted $H(x)$ in the book. Putting in the convergence factors, the Fourier transform is just given by the $x > 0$ part of the transform of unity in Eq. (7), *i.e.* it is given by $(-ik + \epsilon)^{-1}$ with $\epsilon \to 0^+$, which we write as

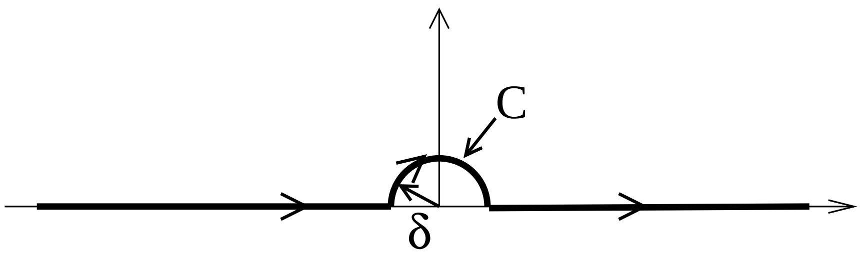
$$\theta(x) \ \xrightarrow{FT} \ \frac{i}{k + i\epsilon} . \tag{32}$$

This equation is to be understood in the same sense as Eq. (17), which is described in the two "bullets" after that equation.

To determine the inverse transformation, it will be convenient, for now, to multiply the Fourier transform of the θ function by a smooth function $u(k)$, i.e. we calculate

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{u(k)e^{-ikx}}{k + i\epsilon} \, dk. \tag{33}$$

The small imaginary part in the denominator is necessary for making this integral well defined. Since the integrand has a pole at $-i\epsilon$ and $\epsilon > 0$, the contour passes passes *above* the pole. In the limit of $\epsilon \to 0^+$, the pole is arbitrarily close to the origin and it is convenient to deform the path of integration so it forms a small semicircle of radius δ above the origin as shown.



We will take $\delta \to 0$ (though $\delta \gg \epsilon$). Hence, for $\epsilon \to 0$

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{u(k)\, dk}{k + i\epsilon} = \lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} \frac{u(k)\, dk}{k} + \int_{\delta}^{\infty} \frac{u(k)\, dk}{k} \right] + \int_C \frac{u(z)\, dz}{z} , \tag{34}$$

where C is the semicircular contour around the origin shown in the above figure. The integral in the square brackets, where we integrate *up to* a small distance below a singularity and *from* the *same* distance above the singularity, is known as the principal value integral. It is denoted by the symbol $\mathcal{P}$, *i.e.*

$$\mathcal{P} \int_{-\infty}^{\infty} f(k)\, dk \equiv \lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} f(k)\, dk + \int_{\delta}^{\infty} f(k)\, dk \right] .$$

Along the semicircle of radius δ, we have $z = \delta e^{i\theta}$ and so, for $\delta \to 0$,

$$\int_C \frac{u(z)\,dz}{z} = \int_\pi^0 u(\delta e^{i\theta})\,\frac{i\delta e^{i\theta}\,d\theta}{\delta e^{i\theta}} = iu(0)\int_\pi^0 d\theta = -i\pi u(0).$$

Consequently we can write Eq. (34) as

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{u(k)\,dk}{k+i\epsilon} = \mathcal{P}\int_{-\infty}^{\infty}\frac{u(k)\,dk}{k} - i\pi u(0). \tag{35}$$

It is frequently useful to forget about the smooth function $u(k)$ and the integration, and write (with $\epsilon \to 0^+$ assumed)

$$\boxed{\frac{1}{k+i\epsilon} = \mathcal{P}\left(\frac{1}{k}\right) - i\pi\delta(k)\,.} \tag{36}$$

Similarly, we find

$$\frac{1}{k-i\epsilon} = \mathcal{P}\left(\frac{1}{k}\right) + i\pi\delta(k)\,. \tag{37}$$

It follows from Eqs. (36) and (32) that the Fourier transform of $\theta(x)$ is given by

$$\boxed{\theta(x) \;\xrightarrow{FT}\; iP\left(\frac{1}{k}\right) + \pi\delta(k)\,.} \tag{38}$$

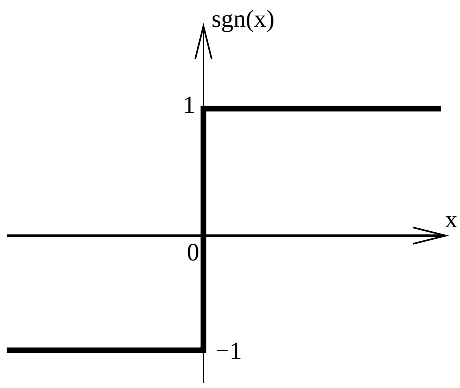
.

Similarly the Fourier transform of $1-\theta(x)$, which takes value 1 for $x<0$ and 0 for $x>0$ is given by the negative x region of the integral in Eq. (7), *i.e.*

$$1-\theta(x) \;\xrightarrow{FT}\; -i\,\mathcal{P}\left(\frac{1}{k}\right) + \pi\delta(k)\,. \tag{39}$$

Note that adding Eqs. (38) and (39) the Fourier transform of unity is found to be $2\pi\delta(k)$ as obtained earlier.

Finally, consider the Fourier transform of the sign function



$$\text{sgn}(x) = \left\{ \begin{array}{r} -1, \ (x < 0) \\ 1, \ (x > 0) \end{array} \right\} = 2\theta(x) - 1.$$

As we shall see, it is also convenient to define sgn(0) = 0, i.e. equal to the *average* of the values on either side of the discontinuity. The sign function is the *difference* between the results in Eqs. (38) and (39), *i.e.*

$$\boxed{\text{sgn}(x) \xrightarrow{FT} 2i\,\mathcal{P}\left(\frac{1}{k}\right) .} \tag{40}$$

It is instructive to verify that the inverse FT of Eq. (40) gives sgn(x). We have

$$f(x) = \frac{1}{2\pi}\, 2i\,\mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{k}\, dk\,. \tag{41}$$

By symmetry only the imaginary part of the complex exponential contributes so

$$f(x) = \frac{1}{\pi}\mathcal{P} \int_{-\infty}^{\infty} \frac{\sin kx}{k}\, dk\,, \tag{42}$$

where the "principal part" symbol can now be taken away because the singularity at $k = 0$ is removed once the cosine part of the integrand in Eq. (41) is eliminated. Changing variables to $k' = kx$, and noting that if $x < 0$ the order of limits is inverted, we get

$$f(x) = \frac{1}{\pi} \left\{ \begin{array}{ll} \displaystyle\int_{-\infty}^{\infty} \frac{\sin k'}{k'}\, dk', & (x > 0) \\ \displaystyle\int_{\infty}^{-\infty} \frac{\sin k'}{k'}\, dk', & (x < 0) \end{array} \right. = \left\{ \begin{array}{l} 1 \ \ (x > 0) \\ -1 \ \ (x < 0) \end{array} \right. ,$$

i.e. $f(x) = \text{sgn}(x)$ as required, where we used the result that

$$\int_{-\infty}^{\infty} \frac{\sin k}{k}\, dk = \pi\,,$$

which can be obtained by contour integration techniques. Finally, if we set $x = 0$ in Eq. (42) we get $f(0) = 0$, in agreement with the general result that, at a discontinuity, the value obtained by a Fourier transform is the average of the limiting values on either side. The above figure illustrates that this is zero for the sign function.

V. OTHER SINGULAR FOURIER TRANSFORMS

A. Powers of x

One can also regularize the FT of functions which grow with a *power* of x at large x, since $x^n e^{-\epsilon|x|} \to 0$ for any finite n and any non-zero (positive) value of ϵ. Typically the result involves a derivative of the delta function. For example, using the methods of this handout one can show that

$$\boxed{x \xrightarrow{FT} -2\pi i\, \delta'(k)\,.} \tag{43}$$

However, it is not possible to regularize functions which diverge *exponentially* at large x, because this divergence is too strong to be canceled by a regularization factor $e^{-\epsilon|x|}$ in the limit $\epsilon \to 0$. For these problems the related technique of *Laplace transforms*, which *can* treat such functions, may be useful. The Laplace transform is defined by

$$g_L(s) = \int_0^\infty f(x)\, e^{-sx}\, ds\,. \tag{44}$$

Compared with a Fourier transform, this has a real exponential and the integral only runs over positive x. Even if $f(x)$ grows like e^{ax} for $x \to \infty$ with $a > 0$, the integral in Eq. (44) still converges provided $s > a$. The Laplace transform is discussed in Boas, Secs. 8.8 and 8.9.

B. Three-dimensions, the Fourier transform of the Coulomb potential

In physics, the Coulomb potential plays an important role, and we need to be able to Fourier transform it. We shall see that this is also a singular Fourier transform and needs regularization to be well defined. The Coulomb potential is

$$f(\vec{r}) = \frac{1}{r}\,, \tag{45}$$

but we shall need to regularize it by converting it to a "screened" Coulomb potential

$$\boxed{f_\epsilon(\vec{r}) = \frac{e^{-\epsilon r}}{r}\,,} \tag{46}$$

in which we will take the limit $\epsilon \to 0$ at the end. Its three-dimensional Fourier transform is

$$g_\epsilon(\vec{k}) = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \int_0^\infty r^2 dr\, \frac{e^{-\epsilon r}}{r} e^{i\vec{k}\cdot\vec{r}}\,, \tag{47a}$$

$$= 2\pi \int_0^\infty r dr\, e^{-\epsilon r} \int_0^\pi \sin\theta\, e^{ikr\cos\theta}\,, \tag{47b}$$

$$= 2\pi \int_0^\infty r dr\, e^{-\epsilon r} \frac{1}{ikr} \left[-e^{ikr\cos\theta}\right]_0^\pi\,, \tag{47c}$$

$$= 2\pi \int_0^\infty r dr\, e^{-\epsilon r} \frac{\left[e^{ikr} - e^{-ikr}\right]}{ikr}\,, \tag{47d}$$

$$= \frac{4\pi}{k} \int_0^\infty \sin(kr)\, e^{-\epsilon r}\, dr\,. \tag{47e}$$

The ϕ integral is done in Eq. (47b) and the θ integral in Eq. (47c). We see that the last integral, Eq. (47e), is not well defined without the convergence factor $e^{-\epsilon r}$. We evaluate this integral by writing $\sin kr = \mathrm{Im}\, e^{ikr}$ so

$$g_\epsilon(\vec{k}) = \frac{4\pi}{k} \,\mathrm{Im} \int_0^\infty e^{(ik-\epsilon)r}\, dr \tag{48a}$$

$$= \frac{4\pi}{k} \,\mathrm{Im} \left[\frac{e^{(ik-\epsilon)r}}{ik-\epsilon} \right]_0^\infty , \tag{48b}$$

$$= \frac{4\pi}{k} \,\mathrm{Im} \left(\frac{1}{\epsilon - ik} \right) , \tag{48c}$$

$$\boxed{= \frac{4\pi}{k^2+\epsilon^2} \,.} \tag{48d}$$

The contribution from the upper limit in Eq. (48b) vanishes because of the convergence factor $e^{-\epsilon r}$. The Fourier transform of the screened Coulomb potential in Eq. (46) is therefore equal to the expression in Eq. (48d).

One often sees this result with the limit $\epsilon \to 0$ explicitly taken, i.e.

$$\boxed{\frac{1}{r} \quad \xrightarrow{3d\,FT} \quad \frac{4\pi}{k^2} \,,} \tag{49}$$

but you should realize that this result only makes sense when the Fourier transform is multiplied by some smooth function of $\vec{k}$ and integrated.

As an application we solve Poisson's equation

$$\nabla^2 u(\vec{r}) = f(\vec{r}) \,, \tag{50}$$

where we have to solve for $u(\vec{r})$ for a given function $f(\vec{r})$ on the right hand side. In a real problem u may be the electrostatic potential in which case f is proportional to the charge density. We solve Eq. (50) by Fourier transforming it with respect to x, y and z. As shown in Eq. (5), differentiating with respect to x, say, brings down a factor of $-ik_x$, and so the Fourier transform of $\nabla^2 u$ is $\left[(-ik_x)^2 + (-ik_y)^2 + (-ik_z)^2\right]$ times the Fourier transform of $u(\vec{r})$, which we call here $\tilde{u}(\vec{k})$. In other works the three-dimensional Fourier transform of Eq. (50) is

$$-k^2 \tilde{u}(\vec{k}) = \tilde{f}(\vec{k}) \,, \tag{51}$$

where $\tilde{f}(\vec{k})$ is the Fourier transform of $f(\vec{r})$. Hence, when Fourier transformed, Poisson's equation is no longer a differential equation but rather a algebraic equation with trivial solution

$$\tilde{u}(\vec{k}) = \frac{\tilde{f}(\vec{k})}{k^2} \,. \tag{52}$$

We now have to Fourier transform back to get the solution in real space. Since the right hand side is a product in Fourier space, it is a convolution in real space, as discussed in Sec. III A. Noting from Eq. (49) that the inverse Fourier transform of $1/k^2$ is $1/(4\pi r)$ we have

$$u(\vec{r}) = -\int \frac{f(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r'\,, \tag{53}$$

which students will recognize as the familiar Coulomb's law.

In the handout on Green function methods to solve PDE's we showed that the solution of Poisson's equation is Eq. (53) plus a solution of the corresponding homogeneous equation (i.e. Laplace's equation), see also Boas §13.8. What happened to the solution of Laplace's equation in Eq. (53)? The answer is that the Fourier transform approach gives the solution which vanishes at infinity, see the discussion in Sec. I. It is easy to see that the solution in Eq. (53) satisfies this condition. In physics, the solution which vanishes at infinity is often the one we want. For example, the potential of a charge distribution localized in some region of space is usually defined to vanish at infinity. That the Fourier transform method *automatically* gives this solution is therefore often an advantage. However, if we want a different boundary condition we have to add an appropriate solution of Laplace's equation to the expression in Eq. (53) in order to satisfy that boundary condition.

VI. SUMMARY

We have discussed several improper Fourier transforms, such as Eqs. (18). (38), (40), (43) and (49). Taken literally, the integrals do not exist and so, as discussed in the text, these equations have to be understood in one of the following senses:

- As a stand-alone equation, in which case the integral in the FT has to be regularized by a convergence factor like $e^{-\epsilon|x|}$.
- Both sides of the equation are multiplied by a smooth function of k and integrated, in which case the convergence factor is unnecessary.

Equation (18) corresponds to the integral representation of the Dirac delta function, Eq. (17), which is very useful as shown in Sec. III.

We also found two additional useful results as byproducts. Firstly, we showed in Eqs. (36) and (37) that

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{54}$$

for $\epsilon \to 0^+$. This equation also needs to be multiplied by a smooth function and integrated in order to make sense. Secondly, we showed how Fourier transforming Poisson's equation very easily gives the solution which vanishes at infinity, namely Eq. (53).

Acknowledgments

I'm grateful to Onuttom Narayan for emphasizing to me the importance of singular Fourier transforms.